算法设计与分析

Lecture 10: Linear Programming

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Classroom Exercise

[2015年陕西卷文-11]

某企业生产甲乙两种产品均需用A, B两种原料, 已知生产1吨每种产品所需原料及每天原料的可用限额如表所示:

	甲	乙	原料限额
A (吨)	3	2	12
B (吨)	1	2	8

如果生产1吨甲乙产品可分别获利3万元, 4万元, 则该企业每天可获得最大利润为( ):

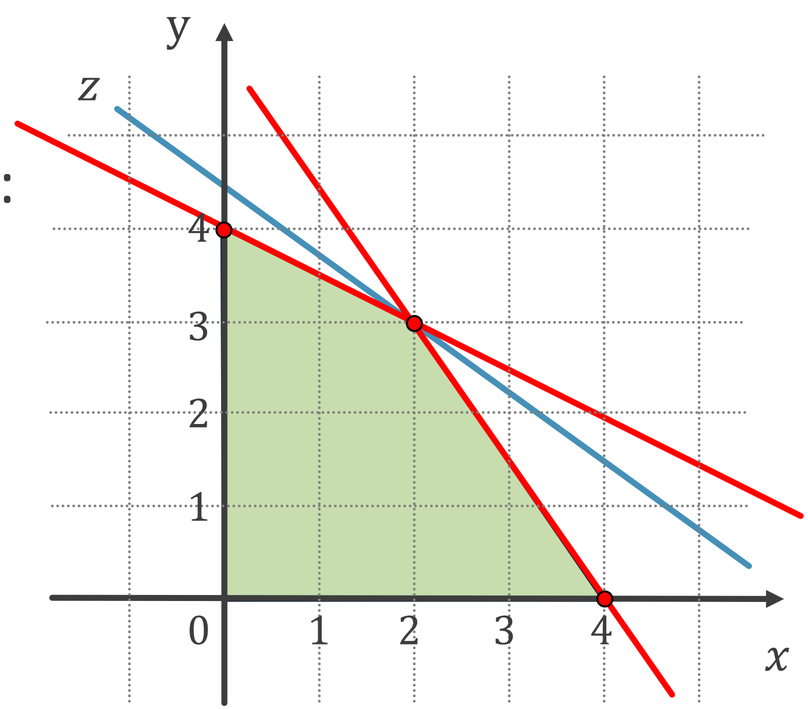
A. 12万元　　B. 16万元　　C. 17万元　　D. 18万元

Classroom Exercise

解:

设该企业每天生产甲乙两种产品分别为x,y吨, 则每天利润为$z=3x+4y$万元, 由题意可列不等式组:

$$\begin{cases} x \geq 0, y \geq 0 \\ 3x+2y \leq 12 \\ x+2y \leq 8 \end{cases}$$



其表示如图阴影部分区域. 当直线$3x+4y-z=0$过点(2,3)时, z取得最大值$z=3\times2+4\times3=18$, 故答案选D.

Classroom Exercise

[命题意图] 本题主要考察线性规划在实际问题中的应用, 建立约束条件和目标函数, 利用数形结合是解决本题的关键.

[方法, 技巧, 规律] 在解决线性规划的应用题时, 可依据以下几个步骤:

(1) 分析题目中相关量的关系, 列出不等式组, 即约束条件和目标函数;

(2) 由约束条件画出可行域;

(3) 分析目标函数z与直线截距之间的关系;

(4) 使用平移直线法求出最优解;

(5) 将最优解还原到现实问题中.

General Linear Programming

- Given a set of real numbers $a_1, a_2, \dots, a_n$ and a set of variables $x_1, x_2, \dots, x_n$, a linear function f on those variables is defined by:

$$f(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n = \sum_{j=1}^{n} a_j x_j$$

- Formally a linear programming (线性规划) problem is the problem of either minimizing or maximizing the linear function f subject to a finite set of linear constraints (线性约束):
  - Linear equality: $f(x_1, \dots, x_n) = b$.
  - Linear inequalities: $f(x_1, \dots, x_n) \le b$, $f(x_1, \dots, x_n) \ge b$.

General Linear Programming

Example 10.1

$$\begin{aligned} \min \quad & -2x_1 + 3x_2 \\ \text{s.t} \quad & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

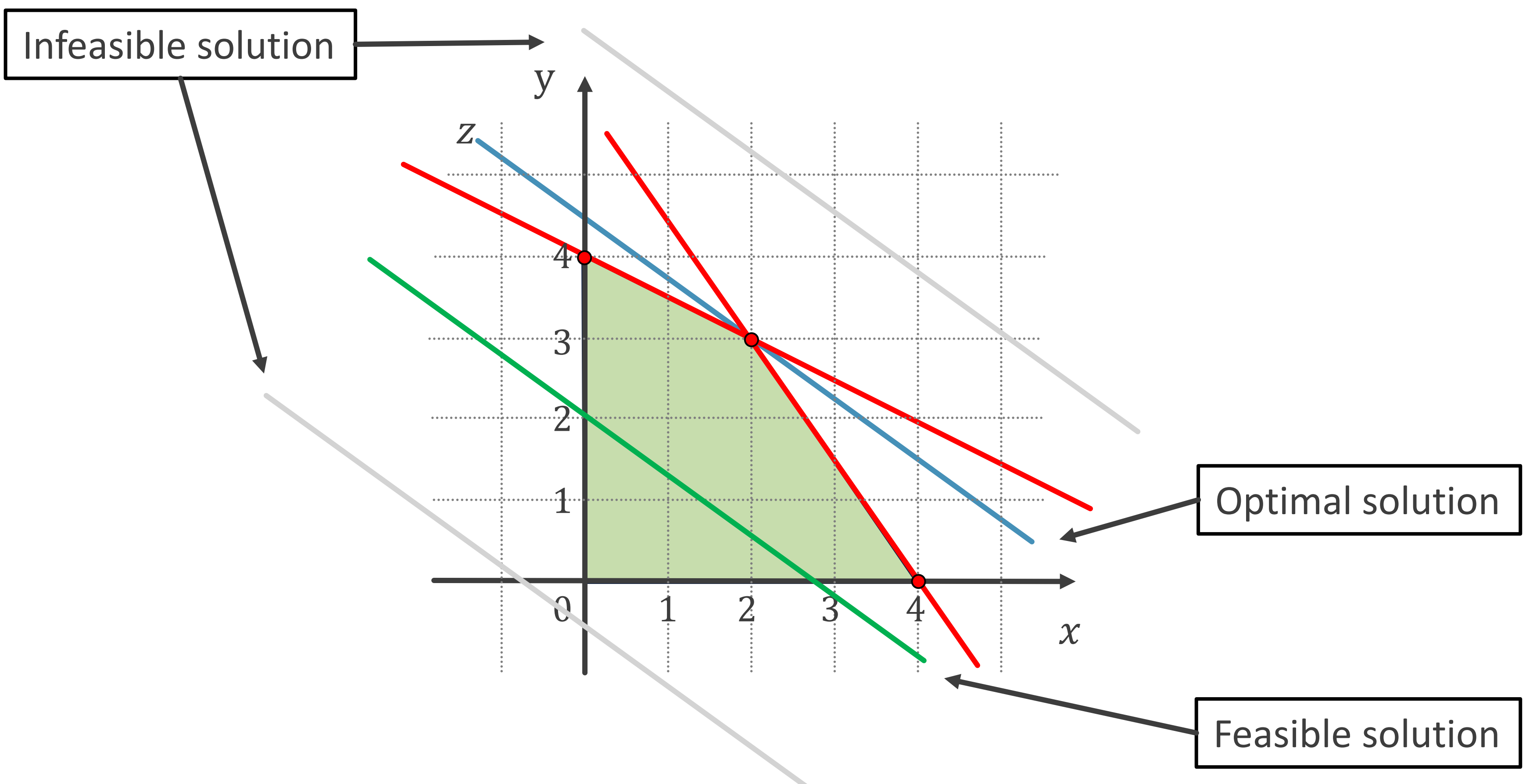
subject to

- We have learned how to solve it in high school. Why do we study here?
- How many variables here? How many constraints here?
  - Can we use high school method to solve linear programming with three variables x_1, x_2, x_3?

Terms for Linear Programming

- Feasible solution (可行解): A solution satisfying all the constraints.
- Infeasible solution (不可行解): A solution not satisfying at least one constraint.
- Objective value (目标值): The goal to maximize or minimize.
- Optimal solution (最优解): The feasible solution to maximize or minimize the objective value.
- Optimal objective value (最优目标值): The objective value calculated by the optimal solution.
- Unbounded (无界的): A linear programming problem with feasible solution but infinite objective value.

Terms for Linear Programming



Standard Form of Linear Programming

- If we want to use computer to solve linear programming problem, we should first convert this problem into a formatted input.
  - We want to build a standard form for this kind of problem.
- Given n real numbers $c_1, c_2, \dots, c_n$; m real numbers $b_1, b_2, \dots, b_m$; and mn real numbers a_{ij} for $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$. We wish to find n real numbers $x_1, x_2, \dots, x_n$ that

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad a_{ij} x_j \le b_i \quad \text{for } i = 1, \dots, m$$

$$x_j \ge 0 \quad \text{for } j = 1, \dots, n$$

number of constrains

number of variables

Standard Form of Linear Programming

- Given the standard form of a linear programming problem:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, \dots, m \\
& x_j \ge 0 \quad \text{for } j = 1, \dots, n
\end{aligned}
$$

- We can extract its coefficients to form an input:

$$
\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}
$$

Standard Form of Linear Programming

- Using vectors and matrices

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

we can represent the standard form as:

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

A n-dim vector with all 0

Standard Form Conversion

- If we have an algorithm to solve the standard form, and convert any problem to the standard form, we can use this algorithm to solve any problem.

min not max

$$\begin{aligned} \min\ & -2x_1 + 3x_2 \\ \text{s.t}\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned}$$

= not ≤

where is $x_2 \ge 0$?

Given problem

$$\begin{aligned} \max\ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.}\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, \dots, m \\ & x_j \ge 0 \quad \text{for } j = 1, \dots, n \end{aligned}$$

Standard form

Standard Form Conversion

1. Convert minimization problem to maximization problem.

$$\min \quad f(x_1, \ldots, x_n) \quad \Rightarrow \quad \max \quad -f(x_1, \ldots, x_n)$$

- Example:

$$\begin{aligned} \min \quad & -2x_1 + 3x_2 \\ \text{s.t} \quad & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 2x_1 - 3x_2 \\ \text{s.t} \quad & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned}$$

Standard Form Conversion

2. Convert equality constraint to two inequality constraints.

$$f(x_1, \dots, x_n) = b \quad \Rightarrow \quad \begin{array}{l} f(x_1, \dots, x_n) \le b \\ f(x_1, \dots, x_n) \ge b \end{array}$$

- Example:

$$\begin{aligned} \min \quad & -2x_1 + 3x_2 \\ \text{s.t} \quad & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 2x_1 - 3x_2 \\ \text{s.t} \quad & x_1 + x_2 \le 7 \\ & x_1 + x_2 \ge 7 \\ & x_1 - 2x_2 \le 4 \\ & x_1 \ge 0 \end{aligned}$$

Standard Form Conversion

3. Convert greater-than-or-equal-to constraint to less-than-or-equal-to constraint.

$$f(x_1, \dots, x_n) \geq b \quad \Rightarrow \quad -f(x_1, \dots, x_n) \leq -b$$

- Example:

$$\begin{aligned} \max \quad & 2x_1 - 3x_2 \\ \text{s.t} \quad & x_1 + x_2 \leq 7 \\ & x_1 + x_2 \geq 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 2x_1 - 3x_2 \\ \text{s.t} \quad & x_1 + x_2 \leq 7 \\ & -x_1 - x_2 \leq -7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

Standard Form Conversion

4. If x_j does not have constraint $x_j \geq 0$, introduce two new variables $x_j{}'$ and x_j'', and let:

$$x_j = x_j{}' - x_j''$$

with two more constraints: $x_j' \geq 0$ and $x_j'' \geq 0$.

> Non-negative $x_j{}'$ and x_j'' can make up x_j with any value

- Example:

$$\begin{aligned} \max \quad & 2x_1 - 3x_2 \\ \text{s.t} \quad & x_1 + x_2 \leq 7 \\ & -x_1 - x_2 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

$$x_2 = x_2{}' - x_2''$$

$$\begin{aligned} \max \quad & 2x_1 - 3x_2' + 3x_2'' \\ \text{s.t} \quad & x_1 + x_2' - x_2'' \leq 7 \\ & -x_1 - x_2' + x_2'' \leq -7 \\ & x_1 - 2x_2{}' + 2x_2'' \leq 4 \\ & x_1, x_2{}', x_2'' \geq 0 \end{aligned}$$

Slack Form of Linear Programming

- Standard form is good enough to represent every linear programming problem as $\boldsymbol{c}$, A and $\boldsymbol{b}$.
- However, it is still not good enough for a computer algorithm to solve. We want the constraints to be either equality or non-negative.
- This form is called slack form (松弛形式).

Slack Form Conversion

1. Convert each inequality constraint into a equality constraint and a non-negative constraint with a new slack variable (松弛变元):

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad \Rightarrow \quad \begin{gathered} x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j \\ x_{n+i} \ge 0 \end{gathered}$$

- Example:

$$\begin{aligned} \max \quad & 2x_1 - 3x_2 + 3x_3 \\ \text{s.t} \quad & x_1 + x_2 - x_3 \le 7 \\ & -x_1 - x_2 + x_3 \le -7 \\ & x_1 - 2x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & 2x_1 - 3x_2 + 3x_3 \\ \text{s.t} \quad & x_4 = 7 - (x_1 + x_2 - x_3) \\ & x_5 = -7 - (-x_1 - x_2 + x_3) \\ & x_6 = 4 - (x_1 - 2x_2 + 2x_3) \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{aligned}$$

Slack Form Conversion

- Slack means difference, to measure how much we can still increase.
- We call the variables in the constraints as basic variables (基本变元), and the variables in the objective function as nonbasic variables (非基本变元).
  - Use B and N to store the index of basic and nonbasic variables.
- Example:

$$
\begin{aligned}
\max \quad & 2x_1 - 3x_2 + 3x_3 \\
\text{s.t} \quad & x_4 = 7 - (x_1 + x_2 - x_3) \\
& x_5 = -7 - (-x_1 - x_2 + x_3) \\
& x_6 = 4 - (x_1 - 2x_2 + 2x_3) \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
B &= \{4,5,6\} \\
N &= \{1,2,3\}
\end{aligned}
$$

Slack Form Conversion

2. Ignore maximization and introduce a new variable z and a constant v to represent the objective value.

$$\max \sum_{j \in N} c_j x_j \quad \Rightarrow \quad z = v + \sum_{j \in N} c_j x_j$$

- Example:

$$\begin{aligned} \max \quad & 2x_1 - 3x_2 + 3x_3 \\ \text{s.t} \quad & x_4 = 7 - (x_1 + x_2 - x_3) \\ & x_5 = -7 - (-x_1 - x_2 + x_3) \\ & x_6 = 4 - (x_1 - 2x_2 + 2x_3) \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

$$\begin{aligned} z &= 0 + 2x_1 - 3x_2 + 3x_3 \\ x_4 &= 7 - (x_1 + x_2 - x_3) \\ x_5 &= -7 - (-x_1 - x_2 + x_3) \\ x_6 &= 4 - (x_1 - 2x_2 + 2x_3) \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Slack Form of Linear Programming

- Given the slack form of a linear programming problem:

$$z = 0 + 2x_1 - 3x_2 + 3x_3$$
$$x_4 = 7 - (x_1 + x_2 - x_3)$$
$$x_5 = -7 - (-x_1 - x_2 + x_3)$$
$$x_6 = 4 - (x_1 - 2x_2 + 2x_3)$$

All variables have non-negative constraint and we can ignore here.

- We can extract its coefficients to form an input:

$$B = \{4,5,6\}, N = \{1,2,3\}, v = 0$$
$$\boldsymbol{c} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 7 \\ -7 \\ 4 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Classroom Exercise

Convert the following linear programming problem into slack form and extract the input vectors and matrix.

$$\begin{aligned} \min \quad & -x_1 - 2x_2 \\ \text{s.t} \quad & x_1 + 2x_2 \leq 6 \\ & -3x_1 - 2x_2 \geq -12 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Classroom Exercise

Solution:

$$
\begin{aligned}
z &= 0 + x_1 + 2x_2 \\
x_3 &= 6 - (x_1 + 2x_2) \\
x_4 &= 12 - (3x_1 + 2x_2) \\
x_5 &= 2 - x_2
\end{aligned}
$$

$$B = \{3,4,5\}, N = \{1,2\}, v = 0$$

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 6 \\ 12 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 0 & 1 \end{bmatrix}$$

Simplex Algorithm

- Simplex algorithm (单纯形法) is an efficient algorithm to solve linear programming in polynomial time.
- It follows several steps to iteratively increase the objective value:
  - Set each nonbasic variable to 0, and compute the values of the basic variables from the equality constraints.
  - Choose a nonbasic variable such that if we were to increase that variable's value from 0, then the objective value would increase too.
  - The slack variables help to determine how much we can increase values of nonbasic values without violating any constraints.

Simplex Algorithm

- Given the slack form of a linear programming problem:

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
$$

- In simplex algorithm, we always set nonbasic variables at 0. Therefore, the initial solution is $(0,0,0,30,24,36)$ with $z = 0$.
- How to increase the value of z?

  Choose a non-negative nonbasic variable and increase it as much as possible.

Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Calculate b_i/a_{ie} for selected nonbasic variable e and every basic variable i.

- If we select x_1 to increase and remain $x_2 = x_3 = 0$, how much as most we can increase x_1?

  Compare $30/1$, $24/2$ and $36/4$ and select the minimum one, which can increase x_1 without violating nonnegative constraints.

Simplex Algorithm

- We select x_1 and increase it up to 9, x_1 is not a nonbasic variable any more.
  - We set all nonbasic variables at 0 at each iteration.

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$\Downarrow$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Simplex Algorithm

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{4} - 4x_3 + \frac{x_6}{2}$$

- Replace $x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$ into objective value and other equality constraints.
- x_1 becomes basic variable and x_6 becomes nonbasic variable. The solution in this step is $(9,0,0,21,6,0)$ with $z = 27$.

Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{4} - 4x_3 + \frac{x_6}{2}$$

- Select x_3 to increase. Compare $9/(\frac{1}{2})$, $21/(\frac{5}{2})$, $\frac{6}{4}$ and select the minimum one: $\frac{3}{2}$.
- We get $x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$.

Simplex Algorithm

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{4} - 4x_3 + \frac{x_6}{2}$$

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$
$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$
$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

- Replace $x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$ into objective value and other equality constraints.
- x_3 becomes basic variable and x_5 becomes nonbasic variable. The solution in this step is $(\frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0)$ with $z = \frac{111}{4}$.

Simplex Algorithm

- Given the slack form of a linear programming problem:

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}
\end{aligned}
$$

We don't calculate here because increase x_2 always satisfy the constraint of x_4.

- Select x_2 to increase. Compare $(\frac{33}{4})/(\frac{1}{16})$, $(\frac{3}{2})/(\frac{3}{8})$ and select the minimum one: 4.
- We get $x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$.

Simplex Algorithm

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$
$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$
$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$
$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{5x_5}{2}$$

- Replace $x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$ into objective value and other equality constraints.
- x_2 becomes basic variable and x_3 becomes nonbasic variable. The solution in this step is $(8,4,0,18,0,0)$ with $z = 28$.

Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$
$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{5x_5}{2}$$

- Now, there's no nonbasic variable that we add increase to increase the objective value. The algorithm terminates with solution $x_1 = 8, x_2 = 4$ and optimal objective value 28.

Pseudocode

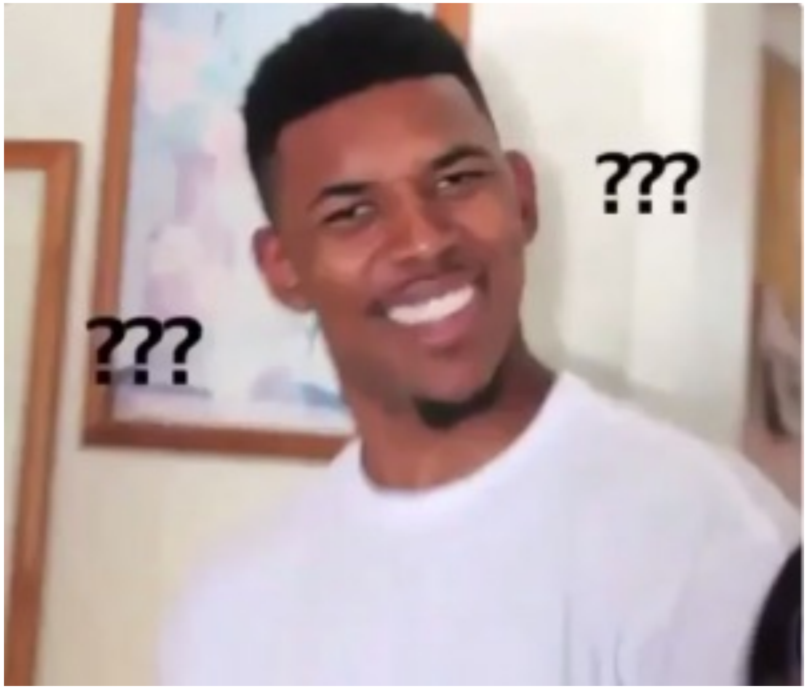
Simplex(A, b, c)

$(N, B, A, b, c, v) \leftarrow$ InitializeSimplex(A, b, c)
while some index $j \in N$ has $c_j > 0$ **do**
  choose an index $e \in N$ for which $c_e > 0$
  for each index $i \in B$ **do**
    if $a_{ie} > 0$ **then** $\mathbf{\Delta}_i \leftarrow b_i / a_{ie}$
    else $\mathbf{\Delta}_i \leftarrow \infty$
  choose an index $l \in B$ that minimizes Δ_i
  if $\Delta_i = \infty$ **then return** "unbounded"
  else $(N, B, A, b, c, v) \leftarrow$ Pivot$(N, B, A,$
      $b, c, v, l, e)$
for $i \leftarrow 1$ **to** n **do**
  if $i \in B$ **then** $\bar{x}_i \leftarrow b_i$
  else $\bar{x}_i \leftarrow 0$
return $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$

Pivot(N, B, A, b, c, v, l, e)

$\hat{b} \leftarrow b_l / a_{le}$
for each $j \in N - \{e\}$ **do** $\hat{a}_{el} \leftarrow a_{lj} / a_{le}$
$\hat{a}_{el} \leftarrow 1 / a_{le}$
for each $i \in B - \{l\}$ **do** $\hat{b}_i \leftarrow b_i - a_{ie}\hat{b}_e$
  for each $j \in N - \{e\}$ **do** $\hat{a}_{ij} \leftarrow a_{ij} - a_{ie}\hat{a}_{ej}$
  $\hat{a}_{il} \leftarrow -a_{ie}\hat{a}_{el}$
$\hat{v} \leftarrow v + c_e\hat{b}_e$
for each $j \in N - \{e\}$ **do** $\hat{c}_j \leftarrow c_j - c_e\hat{a}_{ej}$
$\hat{c}_l \leftarrow -c_e\hat{a}_{el}$
$\hat{N} \leftarrow N - \{e\} \cup \{l\}$
$\hat{B} \leftarrow B - \{l\} \cup \{e\}$
return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{v})$

Simplex Algorithm



What the hell is simplex?

Image source: https://knowyourmeme.com/memes/confused-nick-young

Simplex Algorithm

- Let's go back to the first Gaokao example:

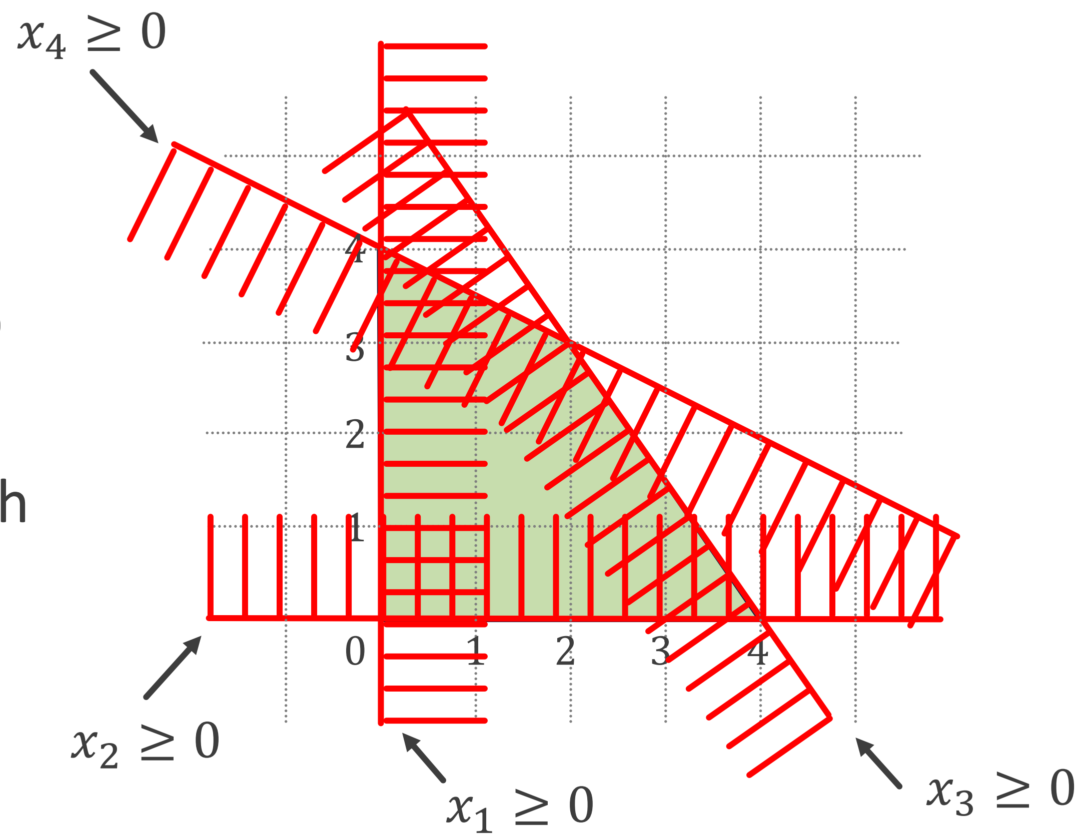
$$
\begin{aligned}
\max \quad & 3x_1 + 4x_2 \\
\text{s.t} \quad & 3x_1 + 2x_2 \leq 12 \\
& x_1 + 2x_2 \leq 8 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

Interpretation of Simplex Algorithm

- Convert it into slack form:

$$
\begin{aligned}
z &= 3x_1 + 4x_2 \\
x_3 &= 12 - (3x_1 + 2x_2) \\
x_4 &= 8 - (x_1 + 2x_2)
\end{aligned}
$$

- Slack form represents each constraint with a variable.

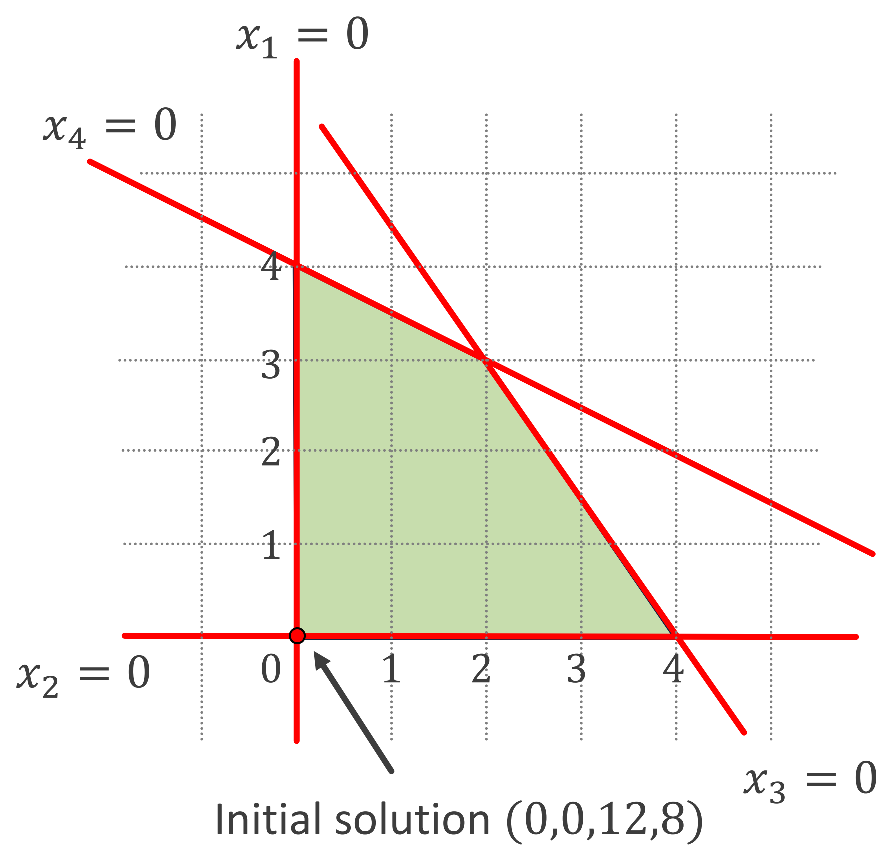


Interpretation of Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 3x_1 + 4x_2$$
$$x_3 = 12 - (3x_1 + 2x_2)$$
$$x_4 = 8 - (x_1 + 2x_2)$$

- We start from the solution (0,0,12,8) with $z = 0$.

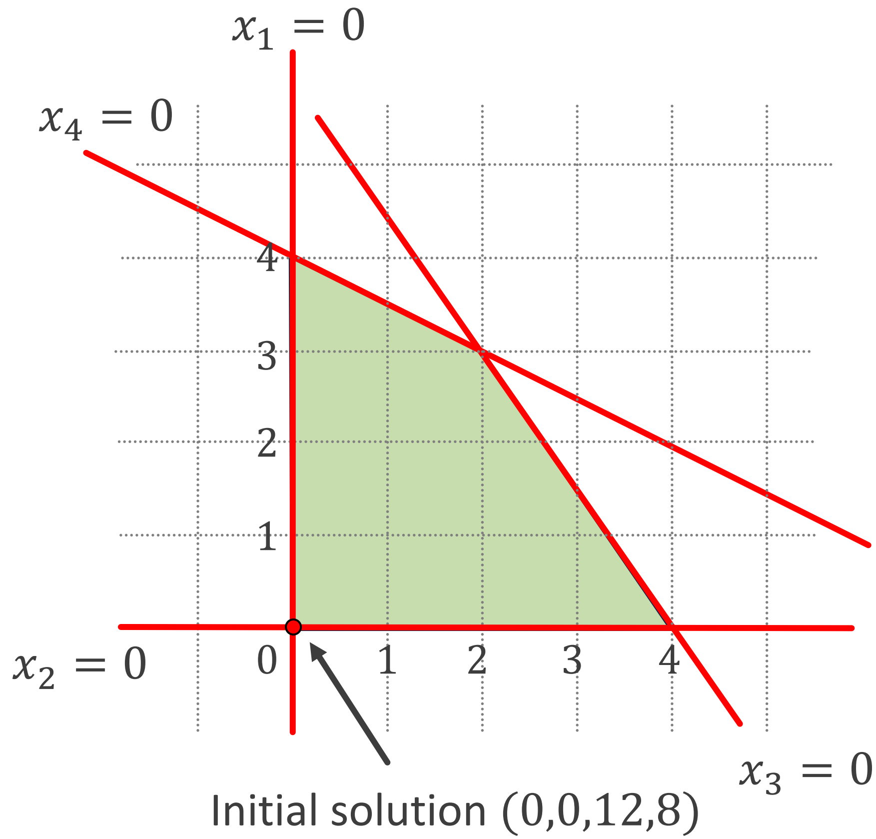


Interpretation of Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 3x_1 + 4x_2$$
$$x_3 = 12 - (3x_1 + 2x_2)$$
$$x_4 = 8 - (x_1 + 2x_2)$$

- Select x_1 to increase. Compare $12/3$, $8/1$ and select the minimum one: 4.
- We get $x_1 = 4 - \frac{2x_2}{3} - \frac{x_3}{3}$.

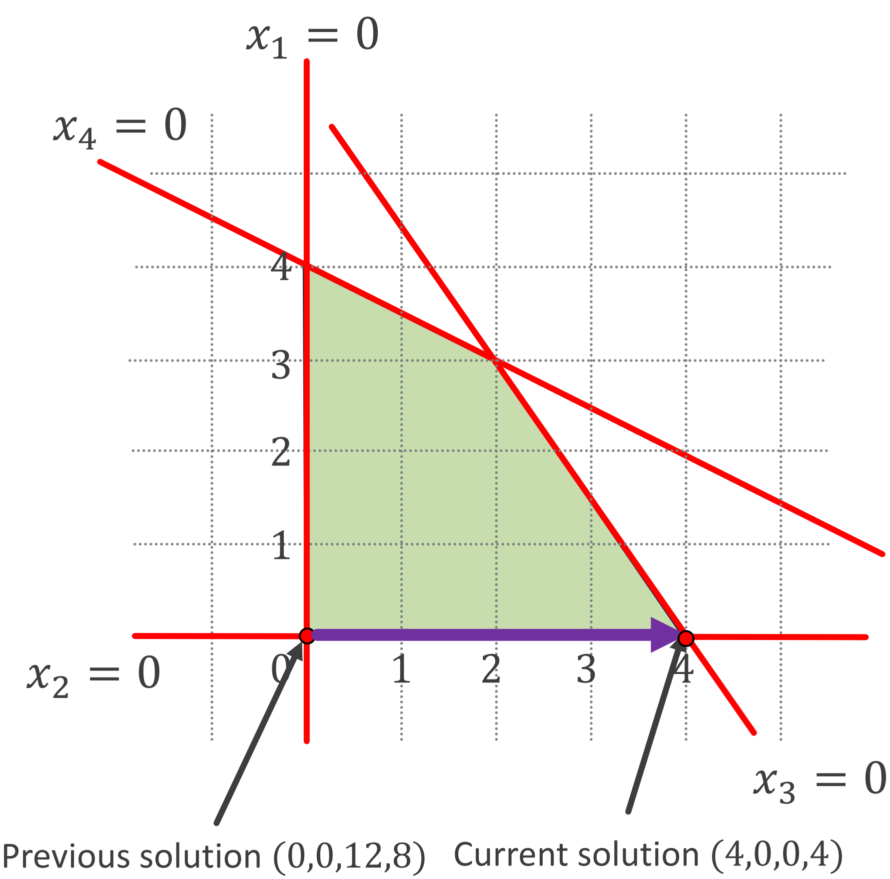


Interpretation of Simplex Algorithm

- We get a new slack form:

$$z = 12 + 2x_2 - x_3$$
$$x_1 = 4 - \frac{2x_2}{3} - \frac{x_3}{3}$$
$$x_4 = 4 - \frac{4x_2}{3} + \frac{x_3}{3}$$

- The objective value increases along x_1.
- Now, x_1 becomes basic variable and x_3 becomes nonbasic variable.

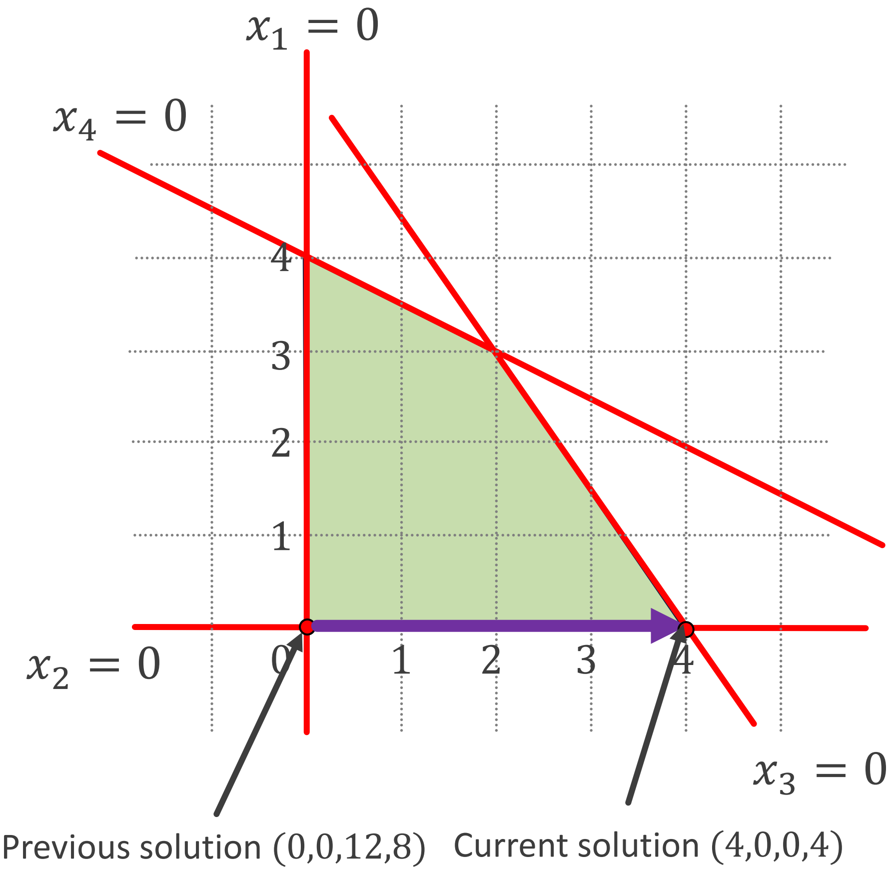


Interpretation of Simplex Algorithm

- We get a new slack form:

$$z = 12 + 2x_2 - x_3$$
$$x_1 = 4 - \frac{2x_2}{3} - \frac{x_3}{3}$$
$$x_4 = 4 - \frac{4x_2}{3} + \frac{x_3}{3}$$

- Now, we can figure out:
  - Nonbasic variable means that the constraint boundary is reached. They are equal to 0.
  - Basic variable means that they the constraint boundary is not reached. There are still spaces to improve.

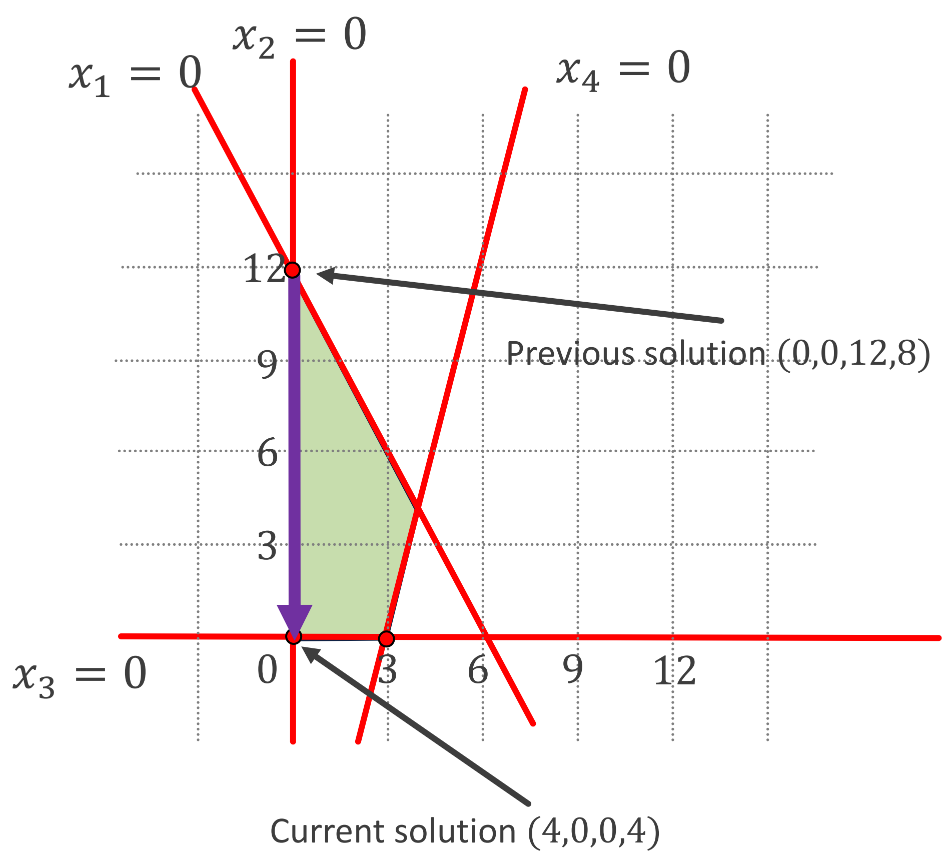


Interpretation of Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 12 + 2x_2 - x_3$$
$$x_1 = 4 - \frac{2x_2}{3} - \frac{x_3}{3}$$
$$x_4 = 4 - \frac{4x_2}{3} + \frac{x_3}{3}$$

- We can reconstruct the figure using nonbasic variables x_2 and x_3 as the new axis.

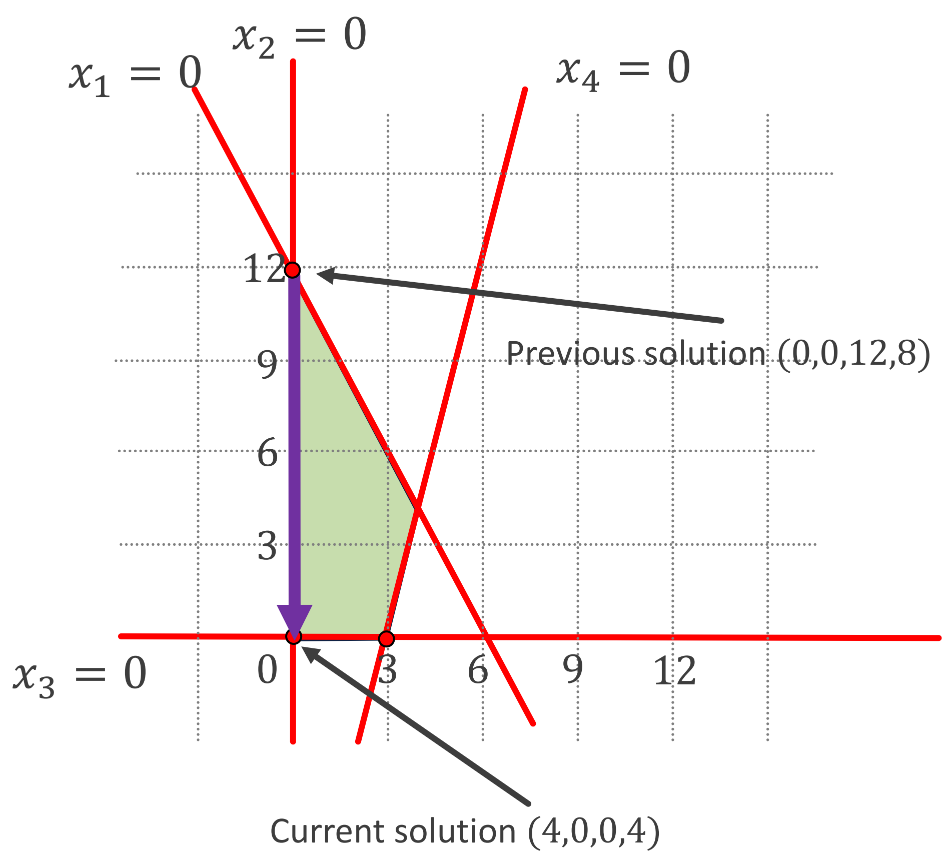


Interpretation of Simplex Algorithm

- Given the slack form of a linear programming problem:

$$z = 12 + 2x_2 - x_3$$
$$x_1 = 4 - \frac{2x_2}{3} - \frac{x_3}{3}$$
$$x_4 = 4 - \frac{4x_2}{3} + \frac{x_3}{3}$$

- Select x_2 to increase. Compare $4/(\frac{2}{3})$, $4/(\frac{4}{3})$ and select the minimum one: 3.
- We get $x_2 = 3 + \frac{x_3}{4} - \frac{3x_4}{4}$.

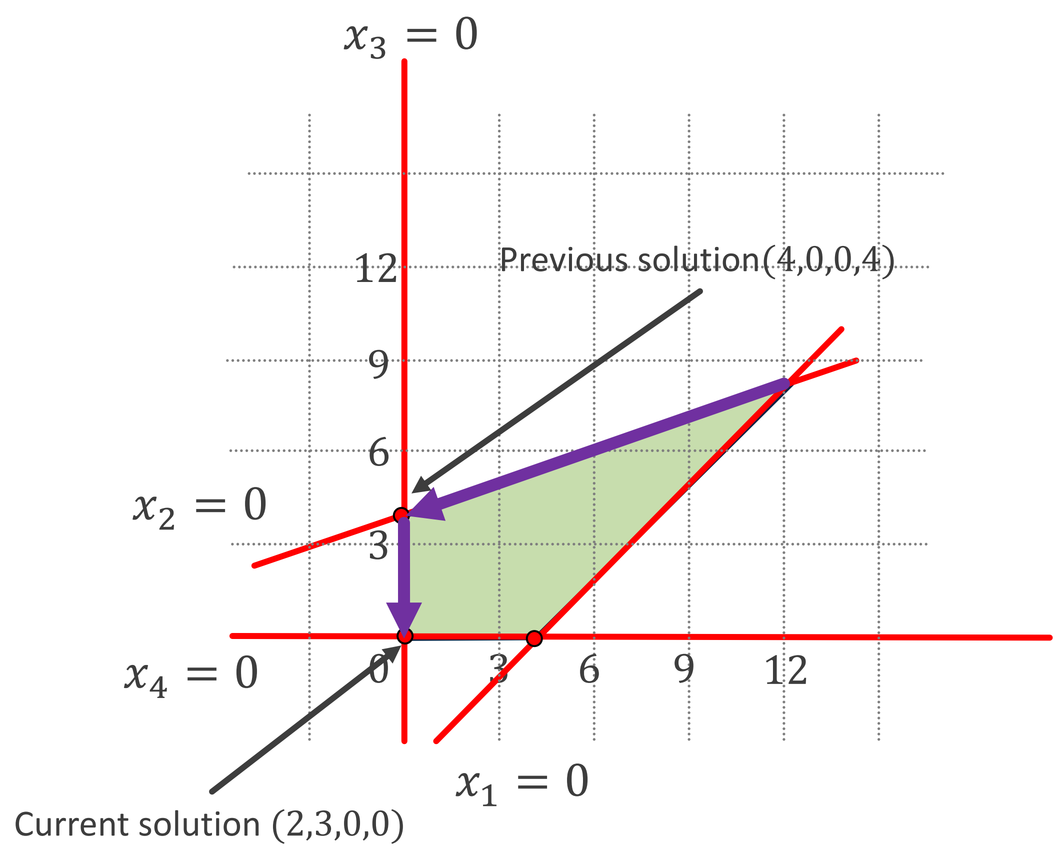


Interpretation of Simplex Algorithm

- We get a new slack form:

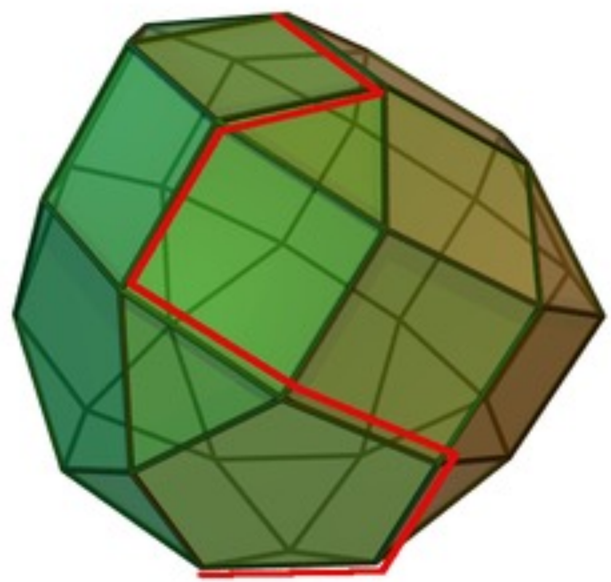
$$z = 18 - \frac{x_3}{2} - \frac{x_4}{2}$$
$$x_1 = 2 - \frac{x_3}{2} + \frac{x_4}{2}$$
$$x_2 = 3 + \frac{x_3}{4} - \frac{3x_4}{4}$$

- The current solution is (2,3,0,0) with $z = 18$.
- The objective value increases along x_2.



Simplex

- In geometry, a simplex is a generalization of the notion of a triangle or tetrahedron (四面体) to arbitrary dimensions.
  - a 0-simplex is a point,
  - a 1-simplex is a line segment,
  - a 2-simplex is a triangle,
  - a 3-simplex is a tetrahedron,
  - a 4-simplex is a 5-cell.



Polyhedron of simplex algorithm in 3D

Image source: https://en.wikipedia.org/wiki/Simplex_algorithm

Classroom Exercise

Use simplex algorithm to solve the following slack form linear programming problem:

$$z = 0 + x_1 + 2x_2$$
$$x_3 = 6 - (x_1 + 2x_2)$$
$$x_4 = 12 - (3x_1 + 2x_2)$$
$$x_5 = 2 - x_2$$

Classroom Exercise

- Given the slack form:

$$
\begin{aligned}
z &= 0 + x_1 + 2x_2 \\
x_3 &= 6 - (x_1 + 2x_2) \\
x_4 &= 12 - (3x_1 + 2x_2) \\
x_5 &= 2 - x_2
\end{aligned}
$$

- Select x_1 and compare 6/1 and 12/3.
- We get: $x_1 = 4 - \frac{2x_2}{3} - \frac{x_4}{3}$.

Classroom Exercise

- Replace $x_1 = 4 - \frac{2x_2}{3} - \frac{x_4}{3}$ in and update the slack form:

$$z = 4 + \frac{4x_2}{3} - \frac{x_4}{3}$$
$$x_1 = 4 - \frac{2x_2}{3} - \frac{x_4}{3}$$
$$x_3 = 2 - \frac{4x_2}{3} + \frac{x_4}{3}$$
$$x_5 = 2 - x_2$$

- Select x_2 and compare $2/(\frac{4}{3})$ and $2/1$.
- We get: $x_2 = \frac{3}{2} - \frac{3x_3}{4} + \frac{x_4}{4}$.

Classroom Exercise

- Replace $x_2 = \frac{3}{2} - \frac{3x_3}{4} + \frac{x_4}{4}$ in and update the slack form:

$$z = 6 - x_3$$
$$x_1 = 3 + \frac{x_3}{2} - \frac{x_4}{2}$$
$$x_2 = \frac{3}{2} - \frac{3x_3}{4} + \frac{x_4}{4}$$
$$x_5 = \frac{1}{2} + \frac{3x_3}{4} - \frac{x_4}{4}$$

- Final solution: $x_1 = 3$, $x_2 = \frac{3}{2}$ with optimal objective value 6.

Conclusion

After this lecture, you should know:

- What is linear programming problem.
- How to convert it into standard form and slack form.
- How to use simplex algorithm to solve linear programming.

Homework

P192-193

10.1

10.3

10.6

10.7

谢谢

有问题欢迎随时跟我讨论